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TWO-PARAMETER OPTIMIZATION OF AN AXIALLY LOADED BEAM ON A FOUNDATION

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A beam of circular cross-section, made of viscoelastic material of Kelvin–Voigt type, is considered. The beam interacts with a foundation of Winkler, Pasternak or Hetényi type. Damping of the foundation is taken into account. The length and volume of the beam are fixed. The beam is symmetric with respect to its center and the radius of the beam is a quadratic function of the co-ordinate. The beam is axially loaded by a non-conservative force $P(t) = P_0 + P_1 \cos \vartheta t$. The ends of the beam are simply supported. Only the first region of instability is considered. The shape of the beam is optimal if the critical value of the amplitude of the oscillating component of the loading is maximal. Numerical examples are presented on graphs and the results are compared with the results obtained for a linearly tapered beam.

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1. INTRODUCTION

The problem of stability and optimization of an axially loaded beam interacting with a foundation with damping has been presented by Foryś [1]. A linearly tapered beam loaded by a force harmonically varying in time was considered.

In the present paper the radius of the beam is a quadratic function of the co-ordinate. Foundations of Winkler, Pasternak or Hetényi type are analyzed. Only the first instability region is considered. Some results of calculations are presented. For the optimal shape of the beam the critical force increases from 1% to 160% in comparison with that of a linearly tapered beam. Results are presented on graphs. The optimal shape of the beam depends on the values of the material constants.

2. FORMULATION OF THE PROBLEM

A straight beam of circular cross-section (see Figure 1) is considered. The undeformed beam axis coincides with the x-axis. The beam is made of viscoelastic material of Kelvin–Voigt type. The length l of the beam and its volume V are fixed. The radius $r(\xi)$ of the cross-section of the beam is given by the formula ($\xi = x/l$)

$$r(\xi) \equiv r_0(\varepsilon_1, \varepsilon_2)\varphi(\xi; \varepsilon_1, \varepsilon_2)$$

$$= \left\{ \begin{array}{ll} r_0\left[\left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{4}\right) + \varepsilon_1\xi + \varepsilon_2\xi^2\right], & \xi \in [0, \tfrac{1}{2}] \\ r_0\left[\left(1 + \frac{\varepsilon_1}{2} + \tfrac{3}{4}\varepsilon_2\right) - (\varepsilon_1 + 2\varepsilon_2)\xi + \varepsilon_2\xi^2\right], & \xi \in [\tfrac{1}{2}, 1] \end{array} \right\}, \qquad (1)$$

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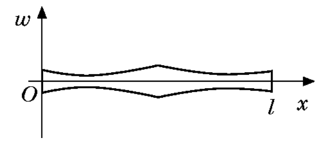

Figure 1. The beam geometry.

where $r_0 \equiv r(\frac{1}{2}) > 0$. The parameters r_0, ε_1 and ε_2 determine the shape of the beam. For a prismatic beam, $\varepsilon_1 = \varepsilon_2 = 0$. For a beam linearly tapered from both ends, one has $\varepsilon_2 = 0$. Because this case has been considered in reference [1], one assumes that $\varepsilon_2 \neq 0$. Also the following assumptions are made: the function $r(\xi)$ is symmetric with respect to $\xi = 0{\cdot}5$ (this is satisfied by equation (1)), $r(0) = r(1) \geqslant 0$ and $r(\xi) > 0$ for $\xi \in (0, 1)$. Because of the last two assumptions, one has the following conditions for ε_1 and ε_2:

$$2\varepsilon_1 + \varepsilon_2 \leqslant 4, \tag{2}$$

$$\{\Delta < 0\} \vee \{\Delta = 0 \wedge \varepsilon_1 \geqslant 0\} \vee \{\Delta = 0 \wedge \varepsilon_1 < -\varepsilon_2\} \vee \{\Delta > 0 \wedge \varepsilon_2 < 0 \wedge \xi_1 \leqslant 0 \wedge \xi_2 > 0{\cdot}5\}$$
$$\vee \{\Delta > 0 \wedge \varepsilon_2 > 0 \wedge \xi_1 \leqslant 0\} \vee \{\Delta > 0 \wedge \varepsilon_2 > 0 \wedge \xi_2 > 0{\cdot}5\}, \tag{3}$$

where $\Delta = (\varepsilon_1 + \varepsilon_2)^2 - 4\varepsilon_2$ and $\xi_{1,2} = (-\varepsilon_1 \pm \sqrt{\Delta})/2\varepsilon_2$ for $\Delta > 0$. The conditions (2) and (3) limit considerations to the physical region in the ε_1–ε_2 plane; it is presented in Figure 2.

Because the volume of the beam is fixed the parameters r_0, ε_1 and ε_2 satisfy the condition

$$r_0 = \sqrt{V/\pi 1(1 - \varepsilon_1/2 - \varepsilon_2/3 + \varepsilon_1^2/12 + \varepsilon_2^2/30 + 5\varepsilon_1\varepsilon_2/48)}. \tag{4}$$

The two independent parameters ε_1 and ε_2 are the optimization parameters.

The beam is axially loaded by a non-conservative force

$$P(t) = P_0 + P_1 \cos \vartheta t, \tag{5}$$

where t is time and P_0, P_1 and ϑ are positive constants. The beam interacts with a foundation of Winkler, Pasternak or Hetényi type with damping.

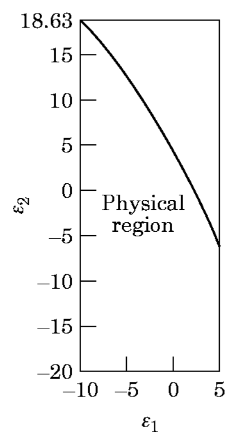

Figure 2. The physical region under consideration.

The equations of the transverse vibrations of the beam on its foundation have the form [2, 1]

$$\frac{1}{\pi^2} f^2(\varepsilon_1, \varepsilon_2) \frac{\partial^2}{\partial \xi^2}\left(\varphi^4 \frac{\partial^2 v}{\partial \xi^2} + \Lambda \varphi^4 \frac{\partial^3 v}{\partial \xi^2\, \partial \tau}\right) + \alpha \frac{\partial^2 v}{\partial \xi^2} + \beta \frac{\partial^2 v}{\partial \xi^2}\cos \theta\tau$$

$$+ \pi^2 f(\varepsilon_1, \varepsilon_2)\varphi^2 \frac{\partial^2 v}{\partial \tau^2} + \pi^2 \kappa v + \pi^2 \gamma \frac{\partial v}{\partial \tau} = 0 \tag{6}$$

for a Winkler model with damping,

$$\frac{1}{\pi^2} f^2(\varepsilon_1, \varepsilon_2) \frac{\partial^2}{\partial \xi^2}\left(\varphi^4 \frac{\partial^2 v}{\partial \xi^2} + \Lambda \varphi^4 \frac{\partial^3 v}{\partial \xi^2\, \partial \tau}\right) + \alpha \frac{\partial^2 v}{\partial \xi^2} + \beta \frac{\partial^2 v}{\partial \xi^2}\cos \theta\tau$$

$$+ \pi^2 f(\varepsilon_1, \varepsilon_2)\varphi^2 \frac{\partial^2 v}{\partial \tau^2} + \pi^2 \kappa v + \pi^2 \gamma \frac{\partial v}{\partial \tau} - \mu \frac{\partial^2 v}{\partial \xi^2} = 0 \tag{7}$$

for a Pasternak model with damping, and

$$\frac{1}{\pi^2} f^2(\varepsilon_1, \varepsilon_2) \frac{\partial^2}{\partial \xi^2}\left(\varphi^4 \frac{\partial^2 v}{\partial \xi^2} + \Lambda \varphi^4 \frac{\partial^3 v}{\partial \xi^2\, \partial \tau}\right) + \alpha \frac{\partial^2 v}{\partial \xi^2} + \beta \frac{\partial^2 v}{\partial \xi^2}\cos \theta\tau$$

$$+ \pi^2 f(\varepsilon_1, \varepsilon_2)\varphi^2 \frac{\partial^2 v}{\partial \tau^2} + \pi^2 \kappa v + \pi^2 \gamma \frac{\partial v}{\partial \tau} + \frac{\delta}{\pi^2}\frac{\partial^4 v}{\partial \xi^4} = 0 \tag{8}$$

for a Hetényi model with damping. In these equations the following dimensionless quantities are introduced [1, 2]:

$$v = w/l, \qquad \tau = (\pi/2l^2)\sqrt{\pi EV/\rho l}t, \qquad \alpha = 4l^4 P_0/\pi EV^2, \qquad \beta = 4l^4 P_1/\pi EV^2,$$

$$\Lambda = (\pi\lambda/2l^2)\sqrt{\pi V/\rho l E}, \qquad \kappa = 4kl^6/\pi^3 EV^2, \qquad \mu = 4Gl^4/\pi EV^2,$$

$$\theta = (2l^2/\pi)\sqrt{\rho l/\pi EV}\vartheta, \qquad \gamma = (2cl^4/\pi^2 EV^2)\sqrt{\pi EV/\rho l},$$

$$\delta = 4\pi D l^4/EV^2, \qquad f(\varepsilon_1, \varepsilon_2) = \left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{3} + \frac{\varepsilon_1^2}{12} + \frac{\varepsilon_2^2}{30} + \frac{5\varepsilon_1\varepsilon_2}{48}\right)^{-1}, \qquad \varphi = \varphi(\xi; \varepsilon_1, \varepsilon_2). \tag{9}$$

Here $w(x, t)$ is the transverse displacement of the cross-section at x at the time t, E is Young's modulus, λ and c are the coefficients of internal and external damping respectively, ρ is the density of the beam, k is the foundation stiffness per unit length, G is the foundation modulus and D is the foundation flexural stiffness.

It is assumed that the two ends of the beam are simply supported:

$$v(0, \tau) = 0, \qquad [\varphi^4(\partial^2 v/\partial \xi^2 + \Lambda\, \partial^3 v/\partial^2 \xi\, \partial \tau)](0, \tau) = 0,$$

$$v(1, \tau) = 0, \qquad [\varphi^4(\partial^2 v/\partial \xi^2 + \Lambda\, \partial^3 v/\partial^2 \xi\, \partial \tau)](1, \tau) = 0. \tag{10}$$

From the equations of motion with the boundary conditions one can determine the first instability region for the beam interacting with its foundation. Shape optimization of the beam consists in finding those values of the parameters ε_1 and ε_2 (from the physical region) for which the value of P_1 (cf., equation (5)) causing the beam's instability is maximal [3, 1].

3. SOLUTION OF THE PROBLEM

The problem is approximately solved by the Galerkin method [1]. Therefore one looks for the solution of equations (6), (7) or (8) in the form

$$v(\xi, \tau) = \sum_{n=1}^{N} q_n(\tau) \sin n\pi\xi \tag{11}$$

and obtains the set of ordinary differential equations for the unknown functions $q_n(\tau)$

$$\sum_{k=1}^{N} (A_{nk}\ddot{q}_k + B_{nk}\dot{q}_k + C_{nk}q_k + D_{nk}q_k \cos\theta\tau) = 0, \qquad n = 1, \ldots, N, \tag{12}$$

where

$$A_{nk} = f(\varepsilon_1, \varepsilon_2) \int_0^1 \varphi^2(\xi; \varepsilon_1, \varepsilon_2) \sin n\pi\xi \sin k\pi\xi \, \mathrm{d}\xi,$$

$$B_{nk} = \gamma\delta_{nk} + \Lambda n^2 k^2 f^2(\varepsilon_1, \varepsilon_2) \int_0^1 \varphi^4(\xi; \varepsilon_1, \varepsilon_2) \sin n\pi\xi \sin k\pi\xi \, \mathrm{d}\xi,$$

$$C_{nk} = (\kappa + \mu n^2 + \delta n^4 - \alpha n^2)\delta_{nk} + n^2 k^2 f^2(\varepsilon_1, \varepsilon_2) \int_0^1 \varphi^4(\xi; \varepsilon_1, \varepsilon_2) \sin n\pi\xi \sin k\pi\xi \, \mathrm{d}\xi,$$

$$D_{nk} = -\beta n^2 \delta_{nk}. \tag{13}$$

Here δ_{nk} is the Kronecker delta. In the relations (13), $\mu \equiv \delta \equiv 0$ for a Winkler foundation, $\delta \equiv 0$ for a Pasternak foundation and $\mu = 0$ for a Hetényi foundation.

Retaining only the first two of equations (12) one determines the boundaries of the first instability region of the beam. The region occurs in the neighbourhood of double the value of the first natural frequency of the beam [4, 5].

The critical value of β (which is proportional to P_1) depends on the values of the optimization parameters ε_1 and ε_2, which determine the shape of the beam. The shape is optimal if the critical value of β is maximal.

4. PARAMETRIC OPTIMIZATION OF THE SHAPE OF THE BEAM

A few examples of numerical calculations for different types of foundation and different values of parameters are presented. The results are illustrated in figures. On the graphs shown in Figures 3–16, the critical value of β as a function of the optimization parameters ε_1 and ε_2, for Winkler, Pasternak and Hetényi models of foundation, respectively, is presented. Optimal values of ε_1 and ε_2 are given and graphs of $r(\xi)$. For comparison the results of reference [1] are sketched in as by dotted lines.

The results show that if the foundation stiffness increases or the internal or external damping increases, the critical force also increases. The results for Pasternak and Hetényi foundations are similar.

The optimal shape of the beam explicitly depends on the values of parameters describing the materials of the beam and the material of foundation. The optimal shape of the beam is not universal—this confirms the results of reference [1].

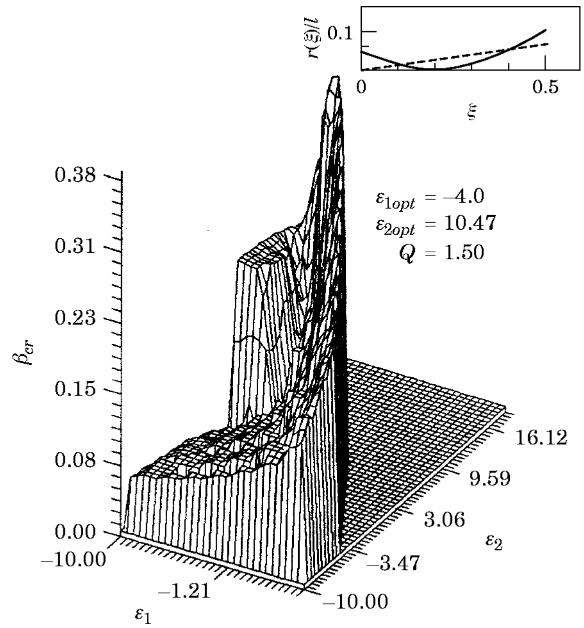

Figure 3. The critical value of β as a function of the optimization parameters ε_1 and ε_2 for a Winkler foundation model; $\kappa = 0{\cdot}1$, $\gamma = 0{\cdot}1$, $\Lambda = 0$, $\alpha = 0{\cdot}5$.

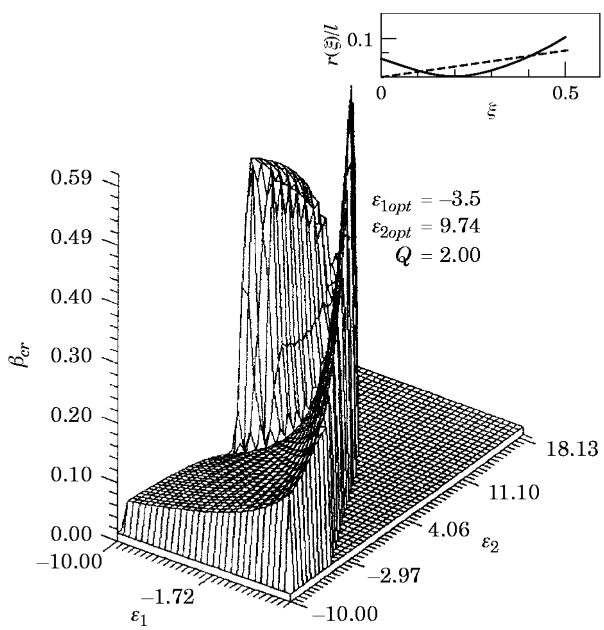

Figure 4. As Figure 3, but $\Lambda = 0{\cdot}01$.

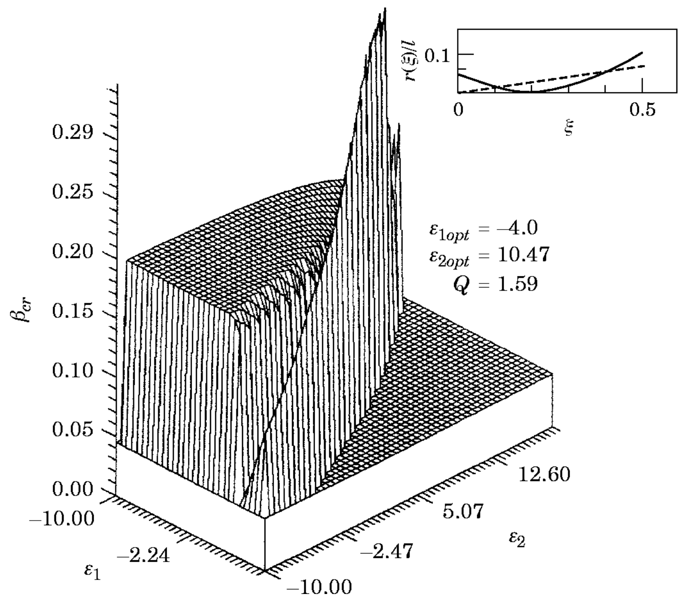

Figure 5. As Figure 3, but $\alpha = 0{\cdot}8$.

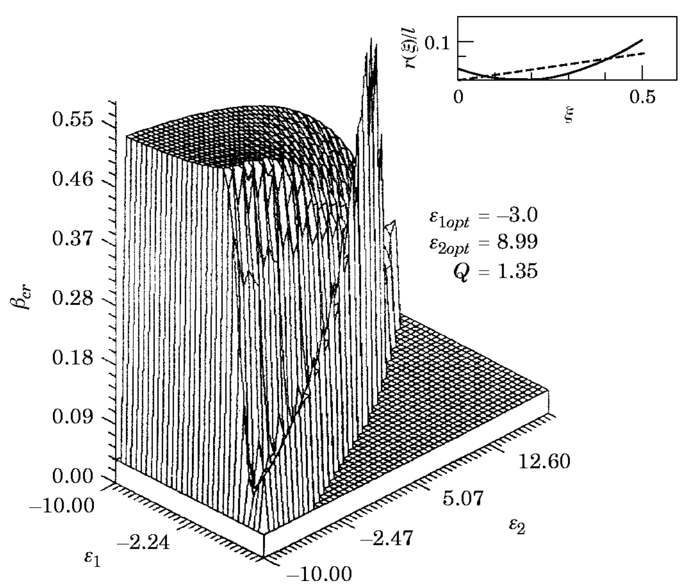

Figure 6. As Figure 5, but $\Lambda = 0{\cdot}01$.

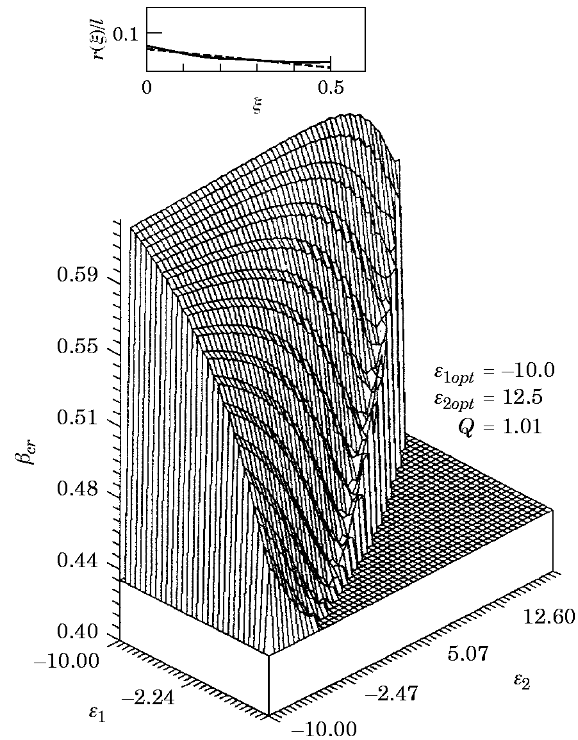

Figure 7. As Figure 3, but $\kappa = 5$.

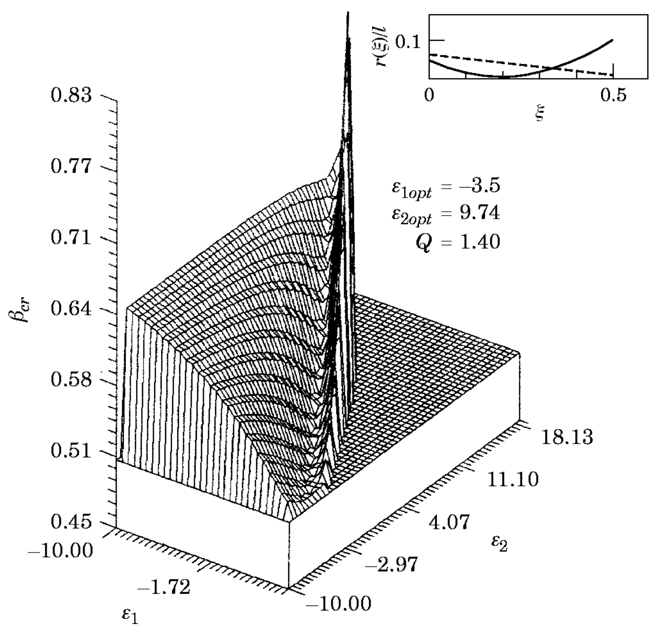

Figure 8. As Figure 7, but $\Lambda = 0{\cdot}01$.

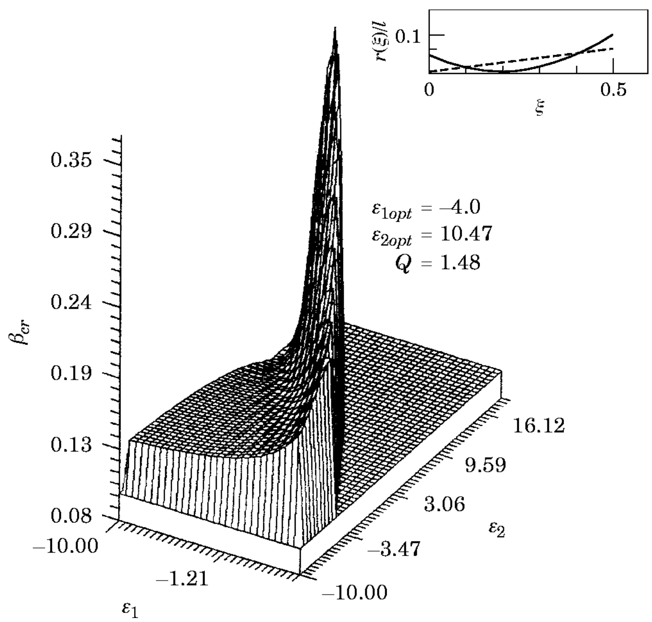

Figure 9. The critical value of β as a function of the optimization parameters ε_1 and ε_2 for a Pasternak foundation model; $\kappa = 0{\cdot}1$, $\gamma = 0{\cdot}1$, $\mu = 0{\cdot}2$, $\Lambda = 0$, $\alpha = 0{\cdot}5$.

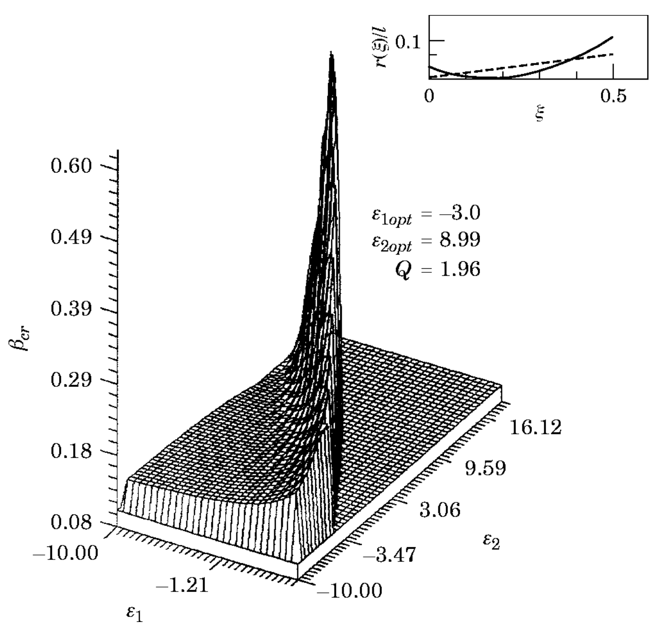

Figure 10. As Figure 9, but $\Lambda = 0{\cdot}01$.

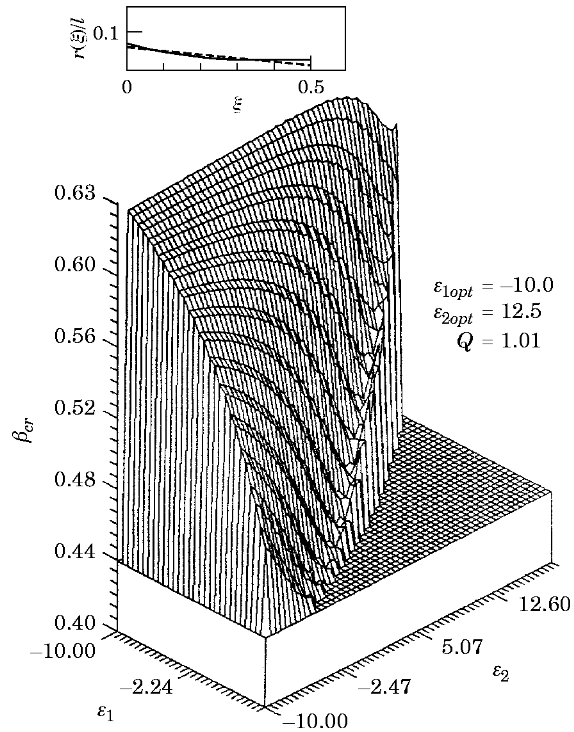

Figure 11. As Figure 9, but $\kappa = 5$.

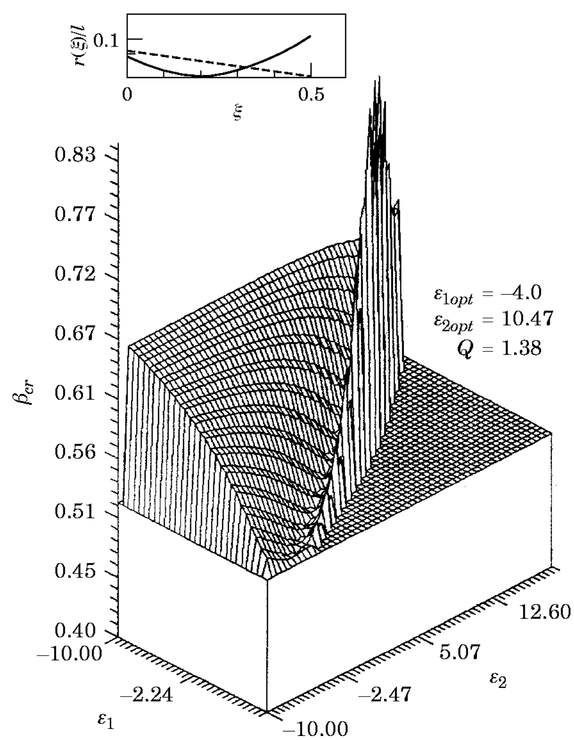

Figure 12. As Figure 11, but $\Lambda = 0{\cdot}01$.

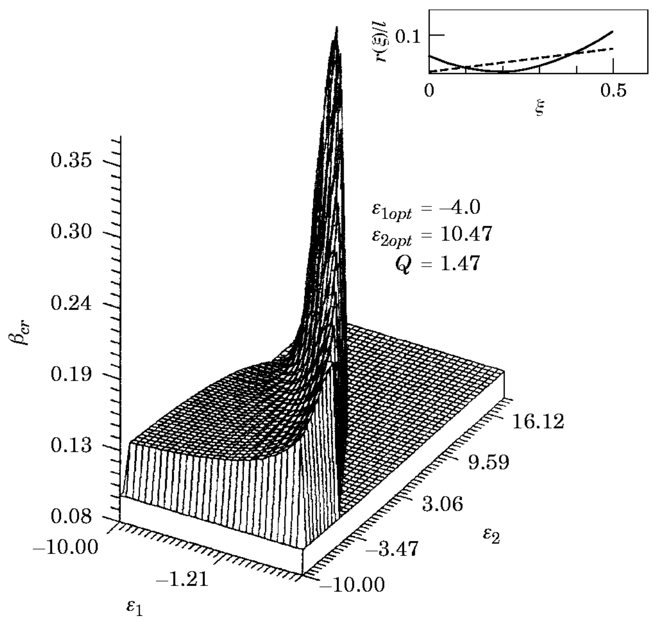

Figure 13. The critical value of β as a function of the optimization parameters ε_1 and ε_2 for a Hetényi foundation model; $\kappa = 0{\cdot}1$, $\gamma = 0{\cdot}1$, $\delta = 0{\cdot}2$, $\Lambda = 0$, $\alpha = 0{\cdot}5$.

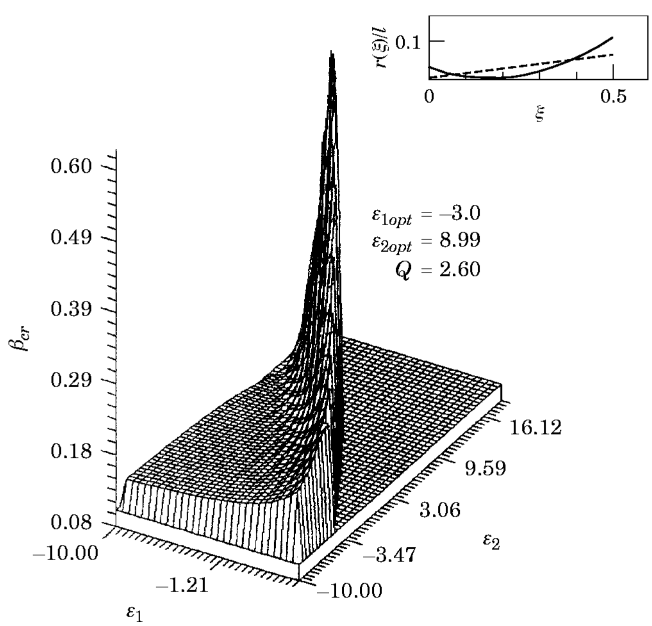

Figure 14. As Figure 13, but $\Lambda = 0{\cdot}01$.

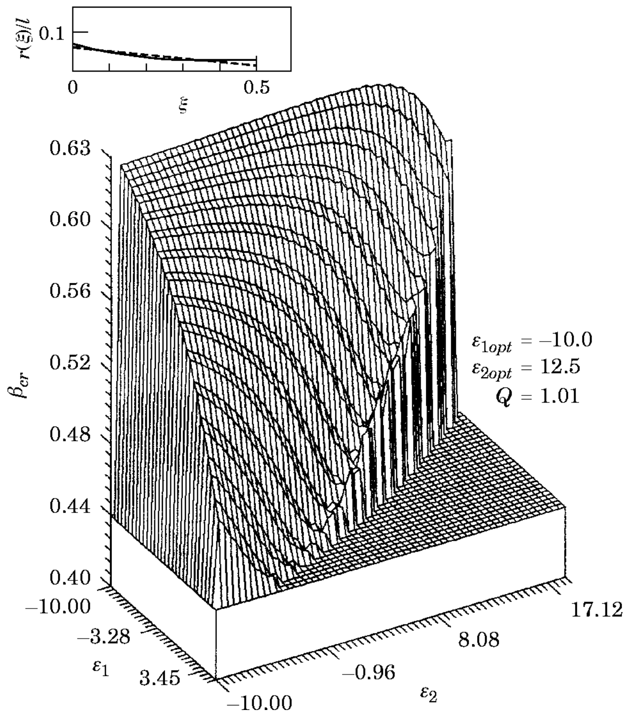

Figure 15. As Figure 13, but $\kappa = 5$.

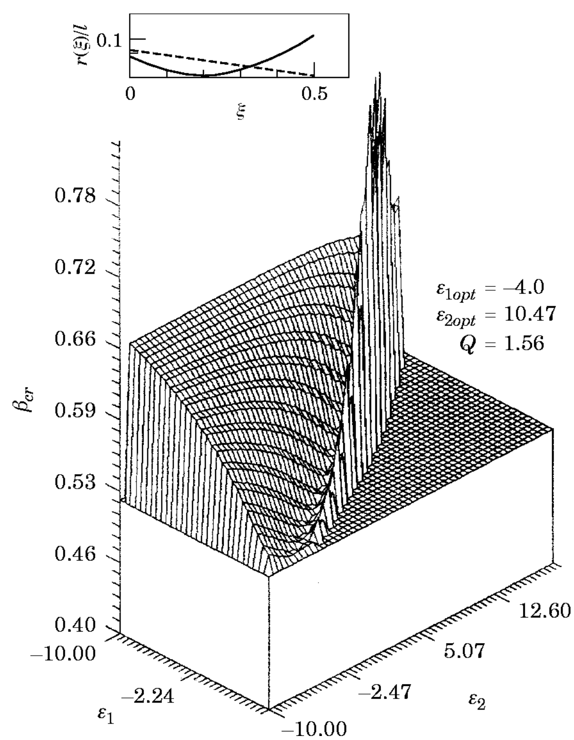

Figure 16. As Figure 15, but $\Lambda = 0{\cdot}01$.

One can introduce the quotient

$$Q = \beta_{opt}/\beta_{opt1}, \tag{14}$$

where β_{opt} is the critical value of β obtained for the optimal shape of the beam and β_{opt1} is the analogous quantity obtained in reference [1]. The values of Q are given in the figures. Because $Q > 1$, the optimal beam is more stable than the optimal beam considered in reference [1].

5. FINAL REMARKS

The parametrical optimal design of an axially loaded, viscoelastic beam has been investigated. The radius of the beam is a quadratic function of the co-ordinate. The beam performs transverse vibration and interacts with a foundation of Winkler, Pasternak or Hetényi type.

The results of the paper suggest that the optimal shape of the beam depends explicitly on the values of the parameters describing the material of the beam and the material of the foundation and that the optimization process is now more effective than for a linearly tapered beam.

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